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Species Doublers as Super Multiplet Partners in Lattice Supersymmetry

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We propose a new lattice superfield formalism in momentum representation which accommodates species doublers of the lattice fermions and their bosonic counterparts as super multiplets. We explicitly show that one dimensional $N = 2$ model with interactions has exact supersymmetry on the lattice for all super charges with lattice momentum. In coordinate representation the finite difference operator is made to satisfy Leibniz rule by introducing a non local product, the "star" product, and the exact lattice supersymmetry is realized. Supersymmetric Ward identities are shown to be satisfied at one loop level.

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1. Introduction

There is a long history of attempts to realize exact supersymmetry on a lattice. See [1] for earlier and recent references. However exact lattice supersymmetry with interactions for full extended supersymmetry has never been realized for gauge fields except for the nilpotent super charge[2, 3, 4]. It has been pointed out that these formulations of lattice SUSY can be essentially reformulated by orbifolding procedure[2, 5].

On the other hand the link approach of lattice SUSY formulation[6] includes the orbifold construction as a specific parameter choice: shift parameter $a = 0$. It was, however, claimed by several authors [7, 8, 5] that an exact SUSY invariance and the gauge invariance are lost for non-vanishing shift parameter case of link approach: $a \neq 0$. Then later it was recognized for non-gauge case that the claim of the exact supersymmetry for link approach is based on the Hopf algebraic symmetry with mild noncommutativity[9].

In finding a possible solution for the difficulties of the link approach, we have found an exact lattice SUSY formulation which includes lattice SUSY algebra in the momentum space. To show the basic ideas and explicit presentation we examine the simplest one-dimensional $N = 2$ supersymmetry model on the lattice. The details of the formulation has already been given in [10]. Here we explain the basic structure of the formulation.

In the coordinate representation of the formulation we introduce a new type of product on which the difference operator surprisingly satisfies Leibniz rule. This new product introduces mild non-locality and thus compatible with a claim of no-go of lattice Leibniz rule for difference operator in [11], where another example of exact lattice SUSY in one dimensional model is given with infinite flavors.

2. Basic ideas

In order to understand the basic structure of lattice SUSY, we first consider the simplest one dimensional model with $N = 1$ symmetry in continuum theory. It is described in terms of a superfield:

$$\Phi(x,\theta) = \varphi(x) + i\theta\psi(x), \tag{2.1}$$

with a supersymmetry charge given by:

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial x}, \qquad Q^2 = i\frac{\partial}{\partial x}. \tag{2.2}$$

This SUSY algebra can be conveniently represented by introducing matrix structure as an internal degree of freedom for super coordinate and derivative:

$$\theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \frac{\partial}{\partial\theta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

which satisfy the following anticommutation relation:

$$\{\frac{\partial}{\partial\theta}\ ,\ \theta\} \ = \ 1. \tag{2.4}$$

Since this representation is not hermitian, hermiticity should be taken care separately.

We may consider this matrix structure as an internal structure of the space time coordinate. With respect to this internal structure the boson φ is considered as a field which commutes with θ and $\frac{\partial}{\partial\theta}$ and the fermion ψ as a field which anticommutes with them. The component fields of boson and fermion with respect to this internal structure has then the following form[12]:

$$\varphi(x) = \begin{pmatrix} \varphi(x) & 0 \\ 0 & \varphi(x) \end{pmatrix}, \qquad \psi(x) = \begin{pmatrix} \psi(x) & 0 \\ 0 & -\psi(x) \end{pmatrix}. \tag{2.5}$$

A super parameter may have the same internal structure as the fermion field.

We now consider to formulate this model on the lattice. In the matrix formulation of fields the coordinate dependence on the lattice can be introduced by diagonal entries of a big matrix as direct product to the internal matrix structure [12]. It is thus very natural to introduce half lattice structure to accommodate the 2×2 matrix internal structure. One can then write a lattice "superfield" corresponding to (2.1) as

$$\text{“}\Phi(x)\text{”} = \varphi(x) + \frac{\sqrt{a}}{2}(-1)^{\frac{2x}{a}}\psi(x), \tag{2.6}$$

where we have introduced a factor $\frac{\sqrt{a}}{2}$ for later convenience and taken away the factor i for hermiticity since the second term is not a product of two Grassmann numbers but only $\psi(x)$ is Grassmann field. In order to accommodate hermiticity in the lattice version of SUSY algebra (2.2) we need to introduce symmetric difference operator to replace the differential operator. With this reason we further need to introduce a quarter lattice and then the superfield "$\Phi(x)$" on the lattice is actually meant as:

$$\Phi(x) = \begin{cases} \varphi(x) & \text{for} \quad x = na/2, \\ \frac{1}{2}a^{1/2}(-1)^{\frac{2x}{a}}\psi(x) & \text{for} \quad x = (2n+1)a/4. \end{cases} \tag{2.7}$$

We now propose lattice supersymmetry transformations as a finite difference over a half lattice spacing $\frac{a}{2}$:

$$\delta\Phi(x) = \alpha a^{-1/2}(-1)^{\frac{2x}{a}}\left[\Phi(x+a/4) - \Phi(x-a/4)\right]. \tag{2.8}$$

By separating $\Phi(x)$ into its component fields according to (2.7) we find:

$$\delta\varphi(x) = \frac{i\alpha}{2}\left[\psi(x+\frac{a}{4}) + \psi(x-\frac{a}{4})\right] \xrightarrow[a\to 0]{} i\alpha\psi(x), \tag{2.9}$$

$$\delta\psi(x) = 2a^{-1}\alpha\left[\varphi(x+\frac{a}{4}) - \varphi(x-\frac{a}{4})\right] \xrightarrow[a\to 0]{} \alpha\frac{\partial\varphi(x)}{\partial x}, \tag{2.10}$$

where x is an even multiple of $a/4$ in (2.9) and an odd one in (2.10)[13]. It is surprising that the half lattice translation together with alternating sign structure (staggered phase) for the lattice superfields generates a correct lattice supersymmetry transformation. We consider that this observation is a key of our formulation.

If we now introduce $N=1$ super charge as $\delta = \alpha Q$, we can show that

$$Q^2\varphi(x) = \frac{i}{a}\left[\varphi(x+a/2) - \varphi(x-a/2)\right], \quad Q^2\psi(x) = \frac{i}{a}\left[\psi(x+a/2) - \psi(x-a/2)\right]. \tag{2.11}$$

This shows that SUSY algebra (2.2) is realized in the lattice level. SUSY transformation on the lattice is half lattice shift with alternating sign structure while super charge square generates single lattice translation.

As we can see in the matrix representation of the component fields $\varphi(x)$ and $\psi(x)$ in (2.5), the same component fields are assigned on the two neighboring half lattice sites. It is natural to double the degrees of freedom for component fields since we have introduced half lattice structure. In this way it is natural to consider $N = 2$ lattice SUSY formulation to accommodate this doubled degrees of freedom. In the following we will show that we can consistently formulate the $D = 1$, $N = 2$ supersymmetry model which contains two bosonic fields and two fermionic fields which are nicely accommodated by species doubler states. It should also be noted that the action for $D = N = 1$ model is fermionic and thus the vacuum is not well defined.

3. $N = 2$ supersymmetry and its lattice realization

The $N = 2$, $D = 1$ supersymmetry algebra has two fermionic generators Q_i satisfying the (anti)commutation relations:

$$Q_1^2 = Q_2^2 = P_t, \qquad \{Q_1, Q_2\} = 0, \qquad [P_t, Q_1] = [P_t, Q_2] = 0, \tag{3.1}$$

where P_t is the generator of translations in the one-dimensional space-time coordinate t. The superfield formulation of (3.1) makes use of two Grassmann odd fermionic coordinates θ_i ($i = 1, 2$), so that the degrees of freedom are given by the superfield expansion:

$$\Phi(t, \theta_1, \theta_2) = \varphi(t) + i\theta_1\psi_1(t) + i\theta_2\psi_2(t) + i\theta_2\theta_1 D(t), \tag{3.2}$$

where ψ_1 and ψ_2 are Majorana fermions. The supersymmetry transformations in terms of the component fields are given by:

$$\delta_j\varphi = i\eta_j\psi_j, \qquad \delta_j\psi_k = \delta_{j,k}\eta_j\partial_t\varphi + \varepsilon_{jk}\eta_j D, \qquad \delta_j D = i\varepsilon_{jk}\eta_j\partial_t\psi_k. \tag{3.3}$$

The supersymmetric action can then be defined in terms of the superfield Φ, for instance with a mass term and a quartic interaction:

$$\int dt d\theta_1 d\theta_2 \left[\frac{1}{2}D_2\Phi D_1\Phi + i\frac{1}{2}m\Phi^2 + i\frac{1}{4}g\Phi^4\right]. \tag{3.4}$$

By integrating over θ_1 and θ_2 one can obtain the action written in terms of the component fields:

$$\begin{aligned} S = \int dt \{ &\frac{1}{2}\left[-(\partial_t\varphi)^2 - D^2 + i\psi_1\partial_t\psi_1 + i\psi_2\partial_t\psi_2\right] \\ &- m(i\psi_1\psi_2 + D\varphi) - g(3i\varphi^2\psi_1\psi_2 + D\varphi^3)\}. \end{aligned} \tag{3.5}$$

According to the discussion of the previous section a lattice of spacing $\frac{a}{2}$ is needed to represent the $N = 1$ supersymmetry on the lattice. In this way however the number of degrees of freedom are doubled. If translations are identified with shifts of a, a field on the lattice admits two translationally invariant configuration: the constant configuration and the configuration constant in absolute value but with alternating sign. Fluctuations around each of these configurations of a lattice field will represent two independent degrees of freedom in the continuum. So the field content of the $N=1$ theory on the lattice, namely one bosonic and one fermionic field, can accommodate the full field content of the $N=2$ theory. We are going to show that the $N=2$ supersymmetry algebra can be represented on the lattice in terms of bosonic field by $\Phi(x)$ with $x=\frac{na}{2}$, and a fermionic field $\Psi(x)$ with $x=\frac{na}{2}+\frac{a}{4}$. As in the $N=1$ case the shift of $\frac{a}{4}$ in the fermionic field with respect to the bosonic one has been introduced to have symmetric finite differences in the supersymmetry transformations and implement hermiticity in a natural way.

One of the two supersymmetry transformations, which we shall denote by δ_1, of the $N=2$ is the same as the one already given on the lattice in the context of the $N=1$ model:

$$\delta_1\Phi(x) = \frac{i\alpha}{2}\left[\Psi(x+\frac{a}{4})+\Psi(x-\frac{a}{4})\right] \qquad x=\frac{na}{2}, \tag{3.6}$$

$$\delta_1\Psi(x) = 2\alpha\left[\Phi(x+\frac{a}{4})-\Phi(x-\frac{a}{4})\right] \qquad x=\frac{na}{2}+\frac{a}{4} \tag{3.7}$$

where $\Phi(x)$ and $\Psi(x)$ are dimensionless, so a rescaling of the fields with powers of a will be needed to make contact with the fields of the continuum theory. In momentum representation the supersymmetry transformations (3.6) and (3.7) read:

$$\delta_1\Phi(p) = i\cos\frac{ap}{4}\alpha\Psi(p)\,, \qquad \delta_1\Psi(p) = -4i\sin\frac{ap}{4}\alpha\Phi(p)\,. \tag{3.8}$$

where for simplicity we used the same letters for fields in momentum representation. $\Phi(p)$ and $\Psi(p)$ satisfy the following periodicity conditions:

$$\Phi(p+\frac{4\pi}{a}) = \Phi(p), \qquad \Psi(p+\frac{4\pi}{a}) = -\Psi(p). \tag{3.9}$$

The commutator of two supersymmetry transformations δ_1 with parameters α and β defines an infinitesimal translation $\delta^t_{\alpha\beta}$ of parameter $\alpha\beta$ on the lattice:

$$\delta^t_{\alpha\beta}F(p) = \delta_{1\beta}\delta_{1\alpha}F(p)-\delta_{1\alpha}\delta_{1\beta}F(p) = 4\sin\frac{ap}{2}\alpha\beta F(p), \tag{3.10}$$

where $F(p)$ stands for either $\Phi(p)$ or $\Psi(p)$. Invariance under (3.10) leads to a non local conservation law where p is replaced by $\sin\frac{ap}{2}$, namely , for a product of fields of momenta $p_1,p_2,...,p_n$:

$$\sin\frac{ap_1}{2}+\sin\frac{ap_2}{2}+\cdots+\sin\frac{ap_n}{2} = 0. \tag{3.11}$$

This conservation law on the lattice was first pointed out by Dondi and Nicolai [14]. In the continuum limit ($ap_i \ll 1$) (3.11) reduces to the standard momentum conservation law and locality is restored. The conservation law (3.11) is not affected if any momentum p_i in it is replaced by $\frac{2\pi}{a}-p_i$ due to the invariance of the sine. The interpretation is clear: in the continuum limit ($ap \ll 1$) $F(p)$ and $F(\frac{2\pi}{a}-p)$ represent fluctuations of momentum p respectively around the vacuum of momentum zero and $\frac{2\pi}{a}$ on the lattice. So the symmetry $p \to \frac{2\pi}{a}-p$ amounts to exchanging the two vacua keeping the physical momentum unchanged.

The correspondence of the lattice fields $\Phi(p)$ and $\Psi(p)$ with the physical fields is

$$\Phi(p) = a^{-\frac{3}{2}}\varphi(p), \qquad \Phi(\frac{2\pi}{a}-p) = -\frac{a^{-\frac{1}{2}}}{4}D(p), \tag{3.12}$$

$$\Psi(p) = a^{-1}\psi_1(p), \qquad \Psi(\frac{2\pi}{a}-p) = ia^{-1}\psi_2(p), \tag{3.13}$$

where p is restricted in (3.12,3.13) to the interval $(-\frac{\pi}{a}, \frac{\pi}{a})$, corresponding to a lattice of spacing a. The rescaling with powers of a keeps track of field dimensionality. With a similar rescaling for the supersymmetry parameter $\alpha = a^{-\frac{1}{2}}\eta$ we obtain for the component fields the following transformations:

$$\delta_1\varphi(p) = i\cos\frac{ap}{4}\eta\,\psi_1(p), \qquad \delta_1 D(p) = \tfrac{4}{a}\sin\tfrac{ap}{4}\eta\,\psi_2(p), \tag{3.14}$$

$$\delta_1\psi_1(p) = -i\frac{4}{a}\sin\frac{ap}{4}\eta\,\varphi(p), \qquad \delta_1\psi_2(p) = \cos\tfrac{ap}{4}\eta D(p). \tag{3.15}$$

which have the correct continuum limit. The component fields in (3.14,3.15) are defined in the interval $-\frac{\pi}{a} < p < \frac{\pi}{a}$, hence in coordinate representation on a lattice with spacing a. However sine and cosine functions in (3.14,3.15) are not periodic of period $\frac{2\pi}{a}$, so that SUSY transformations do not admit a local representation on such lattice. A lattice with $\frac{a}{2}$ spacing is needed for locality.

We have now to identify the second supersymmetry transformation δ_2. In the continuum δ_2 is obtained from δ_1 by replacing everywhere $\psi_1(p)$ with $\psi_2(p)$, and $\psi_2(p)$ with $-\psi_1(p)$. On the lattice this corresponds to:

$$\Psi(p) \longrightarrow -i\Psi(\frac{2\pi}{a} - p). \tag{3.16}$$

By performing this replacement on the supersymmetry transformations (3.8) one obtains the expression for δ_2:

$$\delta_2\Phi(p) = \cos\frac{ap}{4}\alpha\Psi(\frac{2\pi}{a} - p), \qquad \delta_2\Psi(\frac{2\pi}{a} - p) = 4\sin\frac{ap}{4}\alpha\Phi(p)\,. \tag{3.17}$$

The supersymmetry transformation δ_2 satisfies together with δ_1 an $N = 2$ supersymmetry algebra. In fact the commutator of two δ_2 transformations gives an infinitesimal translation as (3.10) and the commutator of a δ_1 and δ_2 transformation vanishes. In terms of the component fields the explicit expression for the δ_2 transformation can be obtained from (3.17):

$$\delta_2\varphi(p) = i\cos\frac{ap}{4}\eta\,\psi_2(p)\ , \qquad \delta_2 D(p) = -\tfrac{4}{a}\sin\tfrac{ap}{4}\eta\,\psi_1(p), \tag{3.18}$$

$$\delta_2\psi_2(p) = -i\frac{4}{a}\sin\frac{ap}{4}\eta\,\varphi(p), \qquad \delta_2\psi_1(p) = -\cos\tfrac{ap}{4}\eta D(p). \tag{3.19}$$

As for δ_1 in the limit $ap \ll 1$ the above transformation coincides, in the momentum space representation, with the one generated by Q_2 in the continuum theory.

The coordinate representation of δ_2 can be obtained directly from (3.18-3.19) by Fourier transform, or from (3.6-3.7) by performing the following substitution:

$$\Psi(x) \longrightarrow (-1)^n\Psi(-x) \qquad x = \frac{na}{2} - \frac{a}{4}, \tag{3.20}$$

which is the same as (3.16) in the coordinate representation. Either way the result is:

$$\delta_2\Phi(x) = \frac{i\alpha}{2}(-1)^n\left[\Psi(-x+\frac{a}{4}) - \Psi(-x-\frac{a}{4})\right] \qquad x = \frac{na}{2}\,, \tag{3.21}$$

$$\delta_2\Psi(x) = 2\alpha(-1)^n\left[\Phi(-x+\frac{a}{4}) - \Phi(-x-\frac{a}{4})\right] \qquad x = \frac{na}{2} + \frac{a}{4}\,. \tag{3.22}$$

It is clear from (3.21) and (3.22) that the supersymmetry transformation δ_2 is local in the coordinate representation only modulo the reflection $x \to -x$. This was already implicit in the correspondence (3.12,3.13) between the lattice fields and the ones of the continuum theory. In fact it is clear from (3.12,3.13) that while for instance $\varphi(x)$ is associated to the fluctuations of $\Phi(x)$ around the constant configuration ($p=0$), the fluctuations of $\Phi(x)$ around the constant configuration with alternating sign ($p=\frac{2\pi}{a}$) correspond in the continuum to $D(-x)$. For fermions this parity change leads to a physical meaning. Since $\psi_2(p) \leftrightarrow \psi_2(x)$ is defined as a species doubler of $\psi_1(p) \leftrightarrow \psi_1(x)$, the chirality of ψ_2 is the opposite of ψ_1. However by the change of $p \to \frac{2\pi}{a} - p$ equivalently $x \to -x$, chirality of ψ_1 and ψ_2 are adjusted to be the same. Thus this bi local nature in the coordinate space may be transferred to a local interpretation.

4. Supersymmetric invariant action

Let us define s_1 and s_2 as the supersymmetry transformations on the lattice without the supersymmetry parameter, namely $\delta_{i,\alpha} = \alpha s_i$. A supersymmetric invariant action on the lattice can be defined in momentum space by giving its n-point term ($n \geq 2$) in the following way:

$$S^{(n)} = g_0^{(n)} a^n \frac{4}{n!} \int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} \frac{dp_1}{2\pi} \cdots \frac{dp_n}{2\pi} \prod_{i=1}^{n} \cos\frac{ap_i}{2} 2\pi\delta\left(\sum_{i=1}^{n} \sin\frac{ap_i}{2}\right) \times s_1 s_2 \Big[\Phi(p_1)\Phi(p_2)\cdots\Phi(p_n)\Big]. \tag{4.1}$$

The sine conservation law enforced by the δ function ensures the invariance of (4.1) under infinitesimal translations (3.10) and the invariance under supersymmetry transformations follow from the supersymmetry algebra. An explicit evaluation of (4.1) gives:

$$\begin{aligned} S_n = g_0^{(n)} a^n \int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} \frac{dp_1}{2\pi} \cdots \frac{dp_n}{2\pi} 2\pi\delta\left(\sum_{i=1}^{n} \sin\frac{ap_i}{2}\right) \times \prod_{i=1}^{n} \cos\frac{ap_i}{2} \\ \times \Big[2\sin^2\frac{ap_1}{4}\,\Phi(p_1)\cdots\Phi(p_n) + \frac{n-1}{4}\,\sin\frac{a(p_1-p_2)}{4}\,\Psi(p_1)\Psi(p_2)\Phi(p_3)\cdots\Phi(p_n)\Big]. \end{aligned} \tag{4.2}$$

For $n > 2$ (4.2) gives a generic interaction term, for $n=2$ it may reproduce both kinetic and mass term. The case $n=2$ is in fact special because only in that case the sine conservation law splits into the two separate conservation laws:

$$p_1 + p_2 = 0, \qquad p_1 - p_2 = \frac{2\pi}{a} \qquad \left(\text{mod. } \frac{4\pi}{a}\right) \tag{4.3}$$

which are linear in the momenta. The delta function in (4.2) can then be replaced without loss of invariance by a superposition with arbitrary coefficients delta functions, namely:

$$a g_0^{(2)} \prod_{i=1}^{2} \cos\frac{ap_i}{2} \delta\left(\sin\frac{ap_1}{2} + \sin\frac{ap_2}{2}\right) \longrightarrow \delta(p_1+p_2) + m_0\delta\left(p_1 - p_2 - \frac{2\pi}{a}\right) \tag{4.4}$$

where m_0 is a free parameter. The first delta function in (4.4) generates the supersymmetric kinetic term, the second delta function the supersymmetric mass term given respectively by:

$$S_{kin} = a\int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} \frac{dp}{2\pi}\left[2\sin^2\frac{ap}{4}\Phi(-p)\Phi(p) - \frac{1}{4}\sin\frac{ap}{2}\Psi(-p)\Psi(p)\right], \tag{4.5}$$

$$S_{mass} = am_0\int_{-\frac{\pi}{a}}^{\frac{3\pi}{a}} \frac{dp}{2\pi}\left[\Phi\left(p+\frac{2\pi}{a}\right)\Phi(p) + \frac{1}{4}\Psi\left(p+\frac{2\pi}{a}\right)\Psi(p)\right]. \tag{4.6}$$

We can now use the correspondence (3.12,3.13) to write (4.5) and (4.6) in terms of the component fields, defined on a $\frac{2\pi}{a}$ interval:

$$S_{kin} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi}\Big[\frac{4}{a^2}(1-\cos\frac{ap}{2})\varphi(-p)\varphi(p)+\frac{1}{4}(1+\cos\frac{ap}{2})D(-p)D(p)- \\ -\frac{1}{a}\sin\frac{ap}{2}\psi_1(-p)\psi_1(p)-\frac{1}{a}\sin\frac{ap}{2}\psi_2(-p)\psi_2(p)\Big]. \tag{4.7}$$

Similarly for the mass term we get:

$$S_{mass} = 2m\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dp}{2\pi}\left[-\varphi(-p)D(p)-i\psi_1(-p)\psi_2(p)\right], \tag{4.8}$$

where m is now the physical mass: $m=\frac{m_0}{a}$. Thanks to the rescaling all fields in (4.7) and (4.8) have the correct canonical dimension, and the continuum limit is smooth. The component fields $\varphi(p)$,$D(p)$,$\psi_1(p)$ and $\psi_2(p)$ are defined for p in the interval $(-\frac{\pi}{a},\frac{\pi}{a})$. This is the Brillouin zone corresponding to a lattice of spacing a, so we could define a lattice with coordinates $\tilde{x}=na$ and the component fields on it as the Fourier transforms of the momentum space components. However the action written in the coordinate $\tilde{x}$ space is non-local, since the finite difference operators appearing in (4.7) are periodic with period $\frac{4\pi}{a}$ and not $\frac{2\pi}{a}$ that would be needed for a local expression on a lattice with spacing a. Instead it is possible to write (4.5) and (4.6) as local actions (modulo $x\to -x$ couplings) on the lattice of spacing $\frac{a}{2}$.

The kinetic term can be written as:

$$S_{kin}=\frac{1}{4}\sum_{x=n\frac{a}{2}}\left[\Phi(x)\left(2\Phi(x)-\Phi(x+\frac{a}{2})-\Phi(x-\frac{a}{2})\right)+\frac{i}{2}\Psi(x+\frac{a}{4})\Psi(x-\frac{a}{4})\right]. \tag{4.9}$$

The coordinate representation for the mass term (4.6) shows instead a coupling between fields in x and $-x$:

$$S_{mass}=\frac{m_0}{2}\sum_{x=n\frac{a}{2}}(-1)^{\frac{2x}{a}}\left[\Phi(-x)\Phi(x)+\frac{i}{4}\Psi(-x-\frac{a}{4})\Psi(x+\frac{a}{4})\right]. \tag{4.10}$$

The bi local structure of (4.10) shows that the extended lattice with spacing $\frac{a}{2}$ has not a straightforward relation to the coordinate space in the continuum limit. This is related to the fact that while the fluctuations of $\Phi(x)$ (resp. $\Psi(x)$) around a constant field configuration are associated to the component field $\varphi(x)$ (resp. $\psi_1(x)$), its fluctuations around $(-1)^{\frac{2x}{a}}$ are associated to $D(-x)$ (resp. $\psi_2(-x)$). In other words the two bosonic (resp. fermionic) components of the superfield are embedded in a single bosonic (resp. fermionic) field on the extended lattice is non trivial and exhibits a bi local structure. Although the extended lattice is not a discrete representation of superspace (bosonic and fermionic fields have to be introduced separately on it) it carries some information about the superspace structure and as such it does not simply map onto the coordinate space in the continuum limit.

For $n>2$ the general invariant expression (4.2) describes interaction terms. The sine conservation law is in these cases intrinsically non-linear in the momenta, and consequently the interaction terms are non-local in coordinate representation. However, as it is shown in the following section, they can be formulated in terms of a non local (but still associative and commutative)product, which we name “star product” , in place of the ordinary field product. As for the kinetic and the mass term the interaction terms can be expressed in terms of the component fields by using the correspondence (3.12, 3.13)and splitting each momentum integration into the two regions $(-\frac{\pi}{a},\frac{\pi}{a})$ and $(\frac{\pi}{a},\frac{3\pi}{a})$. Since each bosonic field $\Phi(p)$ can represent either φ or D, depending on the value of p, the expansion in terms of component fields produces a large number of terms including couplings that do not appear in the continuum limit. Due to the different rescaling of the fields these terms have different powers of a, and by carefully counting the powers of a one finds that in order to have a finite and non vanishing continuum limit for the leading term of (4.1) the physical coupling constant $g^{(n)}$ must be defined as:

$$g^{(n)} = a^{-\frac{n}{2}} g_0^n. \tag{4.11}$$

The naive continuum limit of the lattice theory can then be taken, it is smooth and it reproduces the continuum supersymmetric theory described for instance, for a Φ^4 interaction, in eq. (3.5). This however is not sufficient: invariance under finite translations is violated on the lattice by the sine conservation law. It is then crucial that translational invariance is recovered in the continuum limit. This is not obvious and it requires the analysis of the UV properties of the theory when the continuum limit is taken. The lattice theory described in the previous section in terms of the fields Φ and Ψ is free of ultraviolet divergences. In fact everything in that theory can be written in terms of the dimensionless momentum variables $\tilde{p}_i = ap_i$, which are angular variables with periodicity 4π. Momentum integrations reduce to integrations over trigonometric functions of $\tilde{p}_i$, and ultraviolet divergences never appear. All correlation functions of Φs and Ψs integrations are therefore finite. This however is not enough to ensure that the continuum limit is smooth and that ultraviolet divergences do not appear in the limiting process. The continuum limit in fact involves a rescaling of fields with powers of a, which is singular in the $a \to 0$ limit. At the same time the continuum limit, being a limit where $a \to 0$ keeping the physical momentum fixed, corresponds to a situation where all external momenta $\tilde{p}_i$ are in the neighborhood of one of the vacua, namely at $\tilde{p}_i = 0$ or $\tilde{p}_i = 2\pi$. The limit being a singular one, the ultraviolet behavior has to be checked. This was done in ref. [10] where the lack of UV divergences in the continuum limit was explicitly checked. The recovery of translational invariance in the continuum limit can then be verified, as shown below. Since the conserved quantity on the lattice is not the momentum itself p but $\sin\frac{ap}{2}$ finite translational invariance is explicitly broken at a finite lattice spacing. Indeed, if we denote the component fields by $\phi_A = (\varphi, D, \psi_1, \psi_2)$, the sine conservation law implies that correlation functions are invariant under the transformation:

$$\phi_A(p) \to \exp(il\frac{2}{a}\sin\frac{ap}{2})\phi_A(p) \qquad l : \text{a finite length} \tag{4.12}$$

whereas invariance under finite translation would require the invariance under the transformation

$$\phi_A(p) \to \exp(ilp)\phi_A(p). \tag{4.13}$$

To prove that invariance under finite translations is recovered we need to prove that in the continuum limit (4.12) and (4.13) are equivalent. For an n-point correlation function of ϕ_A, transformation

(4.12) is equivalent to

$$\langle \phi_{A_1}(p_1)\phi_{A_2}(p_2)\dots\phi_{A_n}(p_n)\rangle \to \exp\left(\sum_{i=1}^{n}\frac{2l}{a}\sin\frac{ap}{2}\right)\langle \phi_{A_1}(p_1)\phi_{A_2}(p_2)\dots\phi_{A_n}(p_n)\rangle$$
$$\simeq \left(1-i\frac{a^2 l}{24}\sum_{i=1}^{n}p_i^3\right)\exp\left(il\sum_{i=1}^{n}p_i\right)\langle \phi_{A_1}(p_1)\phi_{A_2}(p_2)\dots\phi_{A_n}(p_n)\rangle, \tag{4.14}$$

where in the last step higher order terms in the expansion of $\sin\frac{ap}{2}$ have been neglected since $ap \to 0$ in the continuum limit. The leading term that breaks translational invariance is then given by the second term in the bracket at the r.h.s. of (4.14). This vanishes as a^2 in the continuum limit if we assume lp_i to be of order 1 so that this term can be neglected as long as no divergence (of order at least $\frac{1}{a^2}$) arises in the correlation function $\langle \phi_{A_1}(p_1)\phi_{A_2}(p_2)\dots\phi_{A_n}(p_n)\rangle$. As discussed before this is not the case, so we can conclude that invariance under finite translations is recovered in the continuum limit.

Finally let us consider invariance under supersymmetry. Invariance under supersymmetry transformations is exact at the finite lattice spacing and it is not spoiled by radiative corrections, which are all finite in the lattice theory. Since the continuum limit is smooth, we expect that exact supersymmetry is preserved also in this limit. This can be confirmed explicitly, as shown in [10], by checking that the corresponding Ward-Takahashi identities (WTi) are satisfied.

5. Coordinate representation

As we have seen in the previous section, momentum conservation on the lattice is the sine momentum conservation. We now try to find out the coordinate counterpart of the corresponding formulation.

With ordinary momentum conservation the product of a field F of momentum p_1 and a field G of momentum p_2 is a composite field $\Phi = F\cdot G$ of momentum $p = p_1 + p_2$, namely the momentum is the additive quantity under product:

$$\Phi(p) \equiv (F\cdot G)(p) = \frac{a}{2\pi}\int dp_1 dp_2 F(p_1)G(p_2)\delta(p-p_1-p_2). \tag{5.1}$$

In coordinate space this amounts to the ordinary local product:

$$\Phi(x) \equiv (F\cdot G)(x) = F(x)G(x). \tag{5.2}$$

On the lattice momentum conservation is replaced by the lattice (sine) momentum conservation (3.11), which means that $\hat{p} = \frac{2}{a}\sin\frac{ap}{2}$ is the additive quantity when taking the product of two fields. In other words the product of a field F of momentum p_1 and a field G of momentum p_2 is a composite field $\Phi = F * G$ of momentum p with $\sin\frac{ap}{2} = \sin\frac{ap_1}{2} + \sin\frac{ap_2}{2}$. This amounts to changing the definition of the "dot" product to that of a "star" product defined in momentum space as:

$$\Phi(p) \equiv (F * G)(p) = \frac{a}{2\pi}\int d\hat{p}_1 d\hat{p}_2 F(p_1)G(p_2)\delta(\hat{p}-\hat{p}_1-\hat{p}_2) \tag{5.3}$$

As we shall see this product is not anymore local in coordinate space but satisfies the Leibniz rule with respect to the symmetric difference operator $\hat{\partial}$. This is easily checked in the momentum representation. In fact acting with the symmetric difference operator corresponds in momentum space to multiplication by $\hat{p} = \frac{2}{a}\sin\frac{ap}{2}$, so that from (5.3) we get:

$$\hat{p}\,\Phi(p) = \frac{a}{2\pi}\int d\hat{p}_1\, d\hat{p}_2\,[\hat{p}_1\, F(p_1)\, G(p_2) + F(p_1)\, \hat{p}_2\, G(p_2)]\,\delta(\hat{p} - \hat{p}_1 - \hat{p}_2). \tag{5.4}$$

Explicit form of the coordinate representation of the star product is given by

$$\begin{aligned}
(F*G)(x) = F(x)*G(x) &= a\int \frac{d\hat{p}}{2\pi}\, e^{-ipx}\,(F*G)(p) \\
&= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} d\tilde{p}\,\cos\tilde{p}\; e^{-ipx}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\tilde{p}_1}{2\pi}\frac{d\tilde{p}_2}{2\pi}\cos\tilde{p}_1\,\cos\tilde{p}_2\int_{-\infty}^{\infty}\frac{d\tau}{2\pi} e^{i\tau(\sin\tilde{p}-\sin\tilde{p}_1-\sin\tilde{p}_2)} \\
&\quad\times\sum_{y,z} e^{i(m\tilde{p}_1 + l\tilde{p}_2)} F(y)G(z) \\
&= \int_{-\infty}^{\infty} d\tau J_{n\pm1}(\tau)\sum_{m,l} J_{m\pm1}(\tau)J_{l\pm1}(\tau)F(y)G(z),
\end{aligned} \tag{5.5}$$

where $\tilde{p} = \frac{ap}{2}$, and $x = \frac{na}{2}, y = \frac{ma}{2}, z = \frac{la}{2}$ should be understood. The lattice delta function is parameterized by τ

$$\delta\left(\frac{2}{a}\sin\tilde{p}_i\right) = \frac{a}{4\pi}\int_{-\infty}^{\infty} d\tau e^{i\tau\sin\tilde{p}_i}. \tag{5.6}$$

$J_n(\tau)$ is a Bessel function defined as

$$J_n(\tau) = \frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} e^{i(n\theta - \tau\sin\theta)} d\theta, \tag{5.7}$$

and we use the following notation:

$$J_{n\pm1}(\tau) = \frac{1}{2}(J_{n+1}(\tau) + J_{n-1}(\tau)). \tag{5.8}$$

It is obvious that the star product is commutative:

$$F(x)*G(x) = G(x)*F(x). \tag{5.9}$$

We can now check how the difference operator acts on the star product of two lattice superfields and find that the difference operator action on a star product indeed satisfies Leibniz rule:

$$\begin{aligned}
& i\hat{\partial}(F(x)*G(x)) \\
&= a\int\frac{d\hat{p}}{2\pi}\, i\hat{\partial}_x\, e^{-ipx}\,(F*G)(p) \\
&= \frac{a^2}{4}\int d\hat{p}\, e^{-ipx}\sum_{y,z}\int\frac{d\hat{p}_1}{2\pi}\frac{d\hat{p}_2}{2\pi}\, e^{ip_1y+ip_2z}\left((i\hat{\partial}_y\, F(y))G(z) \;+\; F(y)\,(i\hat{\partial}_z\, G(z))\right)\,\delta(\hat{p} - \hat{p}_1 - \hat{p}_2) \\
&= (i\hat{\partial}F(x))*G(x) + F(x)*(i\hat{\partial}G(x)).
\end{aligned} \tag{5.10}$$

One can then show that this definition of the $*$-product leads to the vanishing nature of surface terms for $*$-products of several fields:

$$\begin{aligned}
&\sum_x i\hat{\partial}(F(x)*G(x)*H(x)*\cdots) \\
&\quad= \sum_x\left((i\hat{\partial}F(x))*G(x)*H(x) + F(x)*(i\hat{\partial}G(x))*H(x) \,+\, F(x)*G(x)*(i\hat{\partial}H(x)) + \cdots\right) = 0.
\end{aligned} \tag{5.11}$$

It should be noted that if the $*$-product in the above definition is simply replaced by the normal product, the surface terms does not vanish for products of fields more than three fields.

We can now generalize the definition of $*$-product for a product of any fields of the form:

$$F_1(x+b_1)*F_2(x+b_2)*\cdots*F_n(x+b_n) = \int_{-\infty}^{\infty} d\tau J_{n\pm1}(\tau) \sum_{m_1,\cdots,m_n}\left(\prod_{j=1}^{n} J_{m_j\pm1}(\tau)F_j(y_j+b_j)\right), \tag{5.12}$$

where $x = \frac{na}{2}$, $y_j = \frac{m_j a}{2}$ and $J_{n\pm1}(\tau)$ is defined in (5.8).

We can now derive the coordinate representation of the general interaction action $S^{(n)}$. We first note the following relation:

$$2\pi\delta\left(\sum_{j=1}^{n}\sin\frac{ap_j}{2}\right) = \frac{a}{2}\sum_x\int d(\sin\tilde{p})e^{-ipx}\delta\left(\sin\tilde{p}-\sum_{j=1}^{n}\sin\tilde{p}_j\right). \tag{5.13}$$

Then the general interaction action (4.1) can be expressed by the $*$-products:

$$\begin{aligned} S^{(n)} = \frac{4}{n!}g_0^{(n)}\sum_x\Big[&\Big(2\Phi(x)-\Phi(x+\frac{a}{2})-\Phi(x-\frac{a}{2})\Big)*\Phi^{n-1}(x) \\ &+\frac{(n-1)i}{2}\Psi(x+\frac{3a}{4})*\Psi(x+\frac{a}{4})*\Phi^{n-2}(x)\Big], \end{aligned} \tag{5.14}$$

where $\Phi^{n-1}(x)$ is $(n-1)$-th power of $*$-product. The $S^{(2)}$ action in the star products form is equivalent to a sum of both the kinetic terms and the mass terms with fixed coefficient, which include product of local fields and has the following form:

$$S^{(2)} = \sum_x\Big[\Phi(x)*\Big(2\Phi(x)-\Phi(x+\frac{a}{2})-\Phi(x-\frac{a}{2})\Big)+\frac{i}{2}\Psi(x+\frac{3a}{4})*\Psi(x+\frac{a}{4})\Big]. \tag{5.15}$$

It is interesting to recognize that the coordinate representation of the action with star product has almost the same form of the kinetic term of the local action, S_{kin} in (4.9), where the star product is just replaced by the normal product. The arguments of the fermionic lattice superfield in S_{kin} is shifted with $\frac{a}{2}$ from that of (5.15). This is due to the loss of lattice translational invariance in the star product formulation while in eq. (4.9) the translation w.r.t $\frac{a}{2}$ shift is recovered (this is a special feature of bi linear terms) and thus we can obtain the same arguments.

The non local nature of the star product should disappear in the continuum limit. This is however non trivial due to the $p \to \frac{2\pi}{a} - p$ symmetry of the $\sin\frac{ap}{2}$ function and the existence of two translationally invariant vacua at $p = 0$ and $p = \frac{2\pi}{a}$. It was shown by Dondi and Nicolai [14] that in the continuum limit namely at fixed x with $a \to 0$:

$$J_{\frac{2x}{a}}(\tau) \to \delta(\tau - \frac{2x}{a}). \tag{5.16}$$

However in the present context the continuum limit picks up also the configuration at $p = \frac{2\pi}{a}$ and the previous result has to be replaced by:

$$J_{\frac{2x}{a}}(\tau) \to \delta(\tau - \frac{2x}{a}) + (-1)^{\frac{2x}{a}}\delta(\tau + \frac{2x}{a}). \tag{5.17}$$

Thus locality is recovered in the continuum limit, but with an extra coupling of fields in the points x and $-x$ accompanied with the alternating sign factor $(-1)^{\frac{2x}{a}}$. Such remaining non locality disappears when the lattice field Φ and Ψ are reinterpreted in terms of component fields.

6. Conclusion and discussions

We have proposed a new lattice supersymmetry formulation which ensures an exact Lie algebraic supersymmetry invariance on the lattice for all super charges even with interactions. We have introduced bosonic and fermionic lattice superfields which accommodate species doublers as bosonic and fermionic particle fields of super multiplets.

As the simplest model we have explicitly investigated $N = 2$ model in one dimension. The model includes interaction terms and the exact lattice supersymmetry invariance of the action for two supersymmetry charges with lattice momentum are shown explicitly. In the momentum representation of the formulation the standard momentum conservation is replaced by the lattice counterpart of momentum conservation: the sine momentum conservation. The basic lattice structure of this one dimensional model is half lattice spacing structure and the lattice supersymmetry transformation is essentially a half lattice spacing translation. The super coordinate structure and the momentum representation of species doubler fields is hidden implicitly in the alternating sign structure of a half lattice spacing in the coordinate space.

Since the symmetric difference operator does not satisfy Leibniz rule, it was very natural to ask how the supersymmetry algebra be consistent in the coordinate space since super charges satisfy Leibniz rule. In the link approach this problem was avoided by introducing shift nature for super charges. In the current formulation this puzzle is beautifully solved by introducing a new star product of lattice superfields: The difference operator satisfies Leibniz rule on the star products of lattice super fields.

In the definition of the star product in (5.5), non-local summation is introduced with Bessel functions. This non-local feature is not well-behaved non-locality as exponential type but not worse than the inverse distance behaviour. It is expected to be in-between of these two behaviours since Bessel function is integrable. Our claim in this paper includes the statement that exact supersymmetry on the lattice accompanies a non-local behaviour.

Since we have established a new lattice supersymmetry formulation which has exact supersymmetry on the lattice, it would be important to extend the formulation into higher dimensions and to the models with gauge fields. An extension to two dimensions will be given elsewhere.

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